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Sensitivity analysis and optimization corresponding to a degenerate critical point

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Abstract

Sensitivity coefficients of a critical load factor corresponding to a degenerate critical point is shown to be unbounded even for a minor imperfection excluding very restricted case where the imperfection does not have direct effect on the lowest eigenvalue of the stability matrix. This fact leads to serious difficulty in obtaining optimum design under nonlinear stability constraints. The optimum design problem is alternatively formulated with constraint on the lowest eigenvalue of the stability matrix, and the sensitivity formula for the lowest eigenvalue is presented. The existence of a degenerate critical point and accuracy of the sensitivity coefficient are discussed through the example of a four-bar truss.

Keywords: Sensitivity analysis; Stability; Degenerate critical point; Lip singularity; Minor imperfection; Optimization

1. Introduction

Sensitivity analysis of critical load factor of elastic conservative systems has been extensively studied since the pioneering work by Koiter (1945). The purpose of those studies is to present asymptotic formulas for quantitative evaluation of the critical loads with respect to initial imperfection due to, e.g., manufacturing errors and material defects.

There have also been many studies for design sensitivity analysis in the field of optimum design (Haug et al., 1986). Those studies are intended to make analytical and qualitative evaluation of the change in the response due to modification of design variables such as stiffness and nodal location of a skeletal structure. Obviously, imperfection sensitivity and design sensitivity are identical in mathematical sense. Ohsaki and Nakamura (1994) presented an optimum design method based on imperfection sensitivity analysis of limit point loads. For a bifurcation point, it is well known that the sensitivity for a major imperfection is not bounded (Koiter, 1945; Thompson, 1969; Thompson and Hunt, 1973). Therefore, those formulas cannot been used for obtaining optimum designs.

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Structures in practical application, however, usually have symmetry properties, and often reaches an unstable symmetric bifurcation point as the load factor is increased. In this case, antisymmetric imperfection corresponds to a major imperfection that leads to large reduction of the maximum load, and the critical point of an imperfect system is a limit point if the bifurcation points do not coincide. A symmetric imperfection of a symmetric structure, however, is conceived as minor imperfection that does not lead to rapid decrease of the maximum load factor (Roorda, 1968). In this case, the sensitivity coefficients of the bifurcation loads are bounded, and the critical point of the imperfect system remains to be a bifurcation point. In the terminologies of design sensitivity analysis and optimization, a symmetric design modification that is equivalent with a symmetric imperfection is considered as minor imperfection. Ohsaki and Uetani (1996) presented a numerical approach for sensitivity analysis of buckling loads corresponding to a minor imperfection, and applied it for optimizing imperfection sensitive structures (Ohsaki et al., 1998).

The eigenvalues of the tangent stiffness matrix are called stability coefficients in the field of stability analysis. A critical point that has multiple null lowest stability coefficients is called *coincident critical point* (Ho, 1974; Huseyin, 1975; Thompson, 1969). The method by Ohsaki and Uetani (1996) is valid only for a *discrete critical point* where only one stability coefficient vanishes. On the other hand, a *simple critical point* is defined such that the rate of the lowest stability coefficient along the fundamental equilibrium path does not vanish at the discrete critical point (Thompson and Hunt, 1973). The critical point may not be simple even if it is discrete. In this paper, a discrete critical point that is not simple is called *degenerate critical point* which exhibits a *lip singularity* in the terminology of catastrophe theory (Poston and Stewart, 1978; Saunders, 1980).

Existence of a degenerate critical point does not lead to any significant trouble in the sense of instability of the structure, because the lowest stability coefficient immediately increases from zero as the load factor is increased from the critical value. For an optimum design problem, however, a degenerate critical point leads to serious difficulty in the formulation and solution procedure. Ohsaki (2000) presented sensitivity formula with respect to a minor imperfection that can be used even for a coincident critical point including bifurcation and limit points. The formula, however, is not valid for the degenerate critical point.

In this paper, the sensitivity coefficient of a critical load factor corresponding to a degenerate critical point is shown to be unbounded even for a minor imperfection. A new formulation is presented for the optimum design problem under stability constraint. The accuracy of the proposed sensitivity formulation is demonstrated by the example of a four-bar truss.

2. Nonlinear stability analysis

Consider an elastic conservative system subjected to quasi-static proportional loads $\boldsymbol{P} = \Lambda \boldsymbol{P}^0$, where $\boldsymbol{P}^0$ is the base vector and Λ is the load factor. The vector of generalized displacements is denoted by $\boldsymbol{Q} = \{Q_i\}$. The total potential energy is defined in terms of $\boldsymbol{Q}$ and Λ as $\Pi^S(\boldsymbol{Q}, \Lambda)$. Partial differentiation of Π^S with respect to Q_i is written as S_i. Equilibrium condition is derived from the stationary condition of $\Pi^S(\boldsymbol{Q}, \Lambda)$ as

$$S_i = 0 \quad (i = 1, 2, \ldots, f), \tag{1}$$

where f is the number of degree of freedom of the system.

In the following, the summation convention is used only for the subscript. Second partial differentiation of Π^S with respect to the displacements is written as S_{ij}. The rth eigenvalue or stability coefficient $\lambda^r(\Lambda)$ and eigenvector $\Phi^r(\Lambda)$ of the stability matrix $\boldsymbol{S} = [S_{ij}]$ are defined by

$$S_{ij}\phi_j^r = \lambda^r \phi_i^r \quad (i = 1, 2, \ldots, f), \tag{2}$$

where ϕ_i^r is the rth component of Φ^r that is normalized by

$$\phi_i^r \phi_i^r = 1. \tag{3}$$

The critical point is defined by $\lambda^1 = 0$.

Let $\boldsymbol{Q}^F(\Lambda)$ denote the vector of displacements along the fundamental equilibrium path that satisfies Eq. (1). The vector $\boldsymbol{Q}$ may be written in terms of the increment $\boldsymbol{q}$ from $\boldsymbol{Q}^F$ as $\boldsymbol{Q} = \boldsymbol{Q}^F + \boldsymbol{q}$. A new potential energy function Π^W is defined by

$$\Pi^W(\boldsymbol{q}, \Lambda) = \Pi^S(\boldsymbol{Q}^F(\Lambda) + \boldsymbol{q}, \Lambda). \tag{4}$$

Partial differentiation of $\Pi^W(\boldsymbol{q}, \Lambda)$ with respect to q_i is denoted by W_i. Obviously, $S_i = W_i$ and $S_{ij} = W_{ij}$ hold, and the following equation is satisfied:

$$W_{ij}\phi_j^1 = \lambda^1 \phi_i^1. \tag{5}$$

Partial differentiation with respect to Λ is denoted by $(\ )_\Lambda$ along the fundamental equilibrium path. By differentiating Eq. (5) with respect to Λ,

$$W_{ij\Lambda}\phi_j^1 + W_{ij}\phi_{j\Lambda}^1 = \lambda_\Lambda^1 \phi_i^1 + \lambda^1 \phi_{i\Lambda}^1 \tag{6}$$

is derived, where

$$W_\Lambda = S_i Q_{i\Lambda}^F + S_\Lambda. \tag{7}$$

The following equation is to be satisfied at the fundamental equilibrium path by multiplying ϕ_i^1 to Eq. (6), carrying out summation over i, and using Eqs. (3) and (5):

$$W_{ij\Lambda}\phi_j^1 \phi_i^1 = \lambda_\Lambda^1. \tag{8}$$

The critical point where $W_{ij\Lambda}\phi_j^1\phi_i^1 = \lambda_\Lambda^1 \neq 0$ is said to be *simple* (Thompson, 1969). In this paper, the equilibrium state that satisfies $\lambda^1 = \lambda_\Lambda^1 = 0$ is called *degenerate critical point*.

Consider a case where the lowest eigenvalue λ^1 decreases as Λ is increased and λ^1 reaches 0 at the critical load factor Λ^c. For a simple critical point, λ^1 decreases as Λ is further increased from Λ^c. The eigenvalue λ^1, however, might take minimum at $\Lambda = \Lambda^c$ and increase from 0 as Λ is increased from Λ^c as demonstrated in the example of a four-bar truss. In this case, $\lambda_\Lambda^1 = 0$ at $\Lambda = \Lambda^c$, and such a critical point is a degenerate critical point. Variation of λ^1 with respect to Λ is illustrated in Fig. 1. In this case, λ^1 does not have a negative value along the fundamental equilibrium path before reaching another critical point, and existence of the degenerate critical point does not have any serious effect on the stability of the system. Even for an imperfect system, the relation between Λ and a representative displacement component Q is as illustrated in Fig. 2, and the equilibrium state simply makes a detour around the critical point. In the formulation of an optimum design problem, however, the existence of a degenerate critical point has serious effect on the problem formulation and the solution strategy.

Fig. 3 shows the stability boundary with respect to Λ and two imperfection parameters η and μ that correspond to minor and major imperfection, respectively. Note that the region above the boundary is stable. The stability boundary is called bifurcation set in the field of catastrophe. For a system with $\eta = \mu = 0$, a degenerate critical point is reached as Λ is increased. In this case, there is no critical point for $\eta > 0$. If we cut the stability boundary with the plane $\eta = -\eta_0$, where η_0 is a small positive number, we have a boundary curve like a lip as shown in Fig. 3. Although the existence of lip singularity is briefly stated in Saunders (1980) and Thompson and Hunt (1984), characteristics of this singularity have not been extensively discussed.

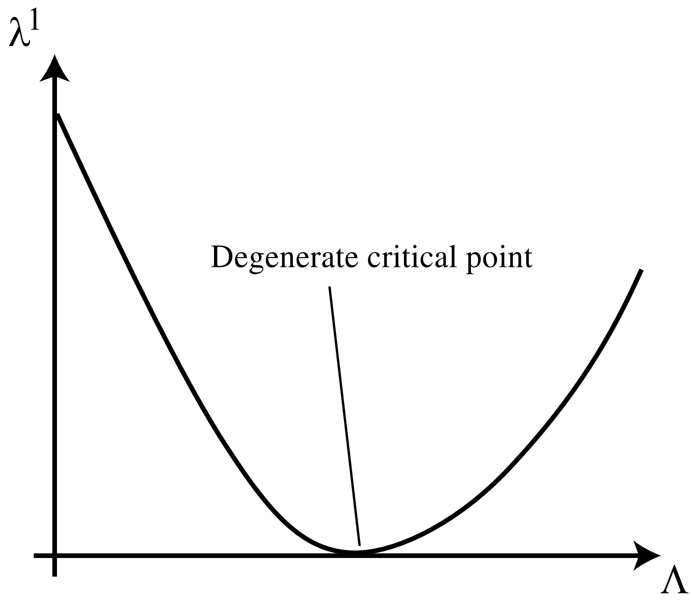

Fig. 1. Variation of lowest eigenvalue with respect to load factor.

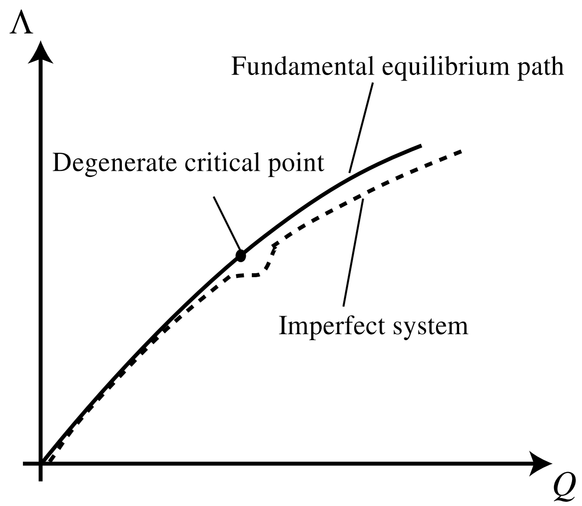

Fig. 2. Relation between load factor and displacement for perfect and imperfect systems.

3. Sensitivity analysis and optimum design problem

Consider a structure whose mechanical properties are defined by the variables such as nodal locations and element stiffnesses, and suppose that the structure has an axis or a plane of symmetry. A symmetric structure subjected to symmetric loads is called *symmetric system* for brevity. Let $\boldsymbol{b}$ denote the vector of

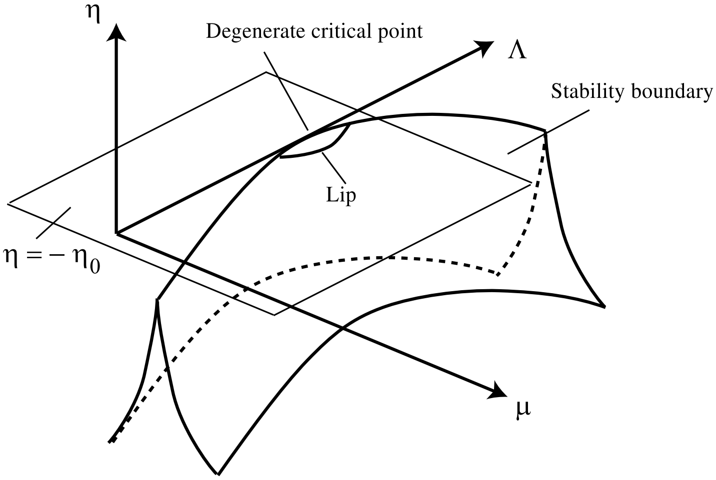

Fig. 3. A degenerate critical point and a lip.

design variables that represent the element stiffness such as the thickness of a plate element and the cross-sectional area of a truss element. Sensitivity analysis is carried out for a symmetric system considering symmetric design modification which is defined by a mode vector $\boldsymbol{b}^s$. A set of symmetric systems in the vicinity of the initial system $\boldsymbol{b}_0$ is defined by using a design parameter ξ as

$$\boldsymbol{b} = \boldsymbol{b}_0 + \xi \boldsymbol{b}^s. \tag{9}$$

Note that $\xi \boldsymbol{b}^s$ may directly correspond to a change in the cross-sectional area of a member, or a coordinate of a node. In this section, difficulties are discussed for formulation of optimum design problem, and sensitivity equations with respect to ξ are presented for the lowest eigenvalue of the stability matrix. Note again that imperfection sensitivity in the field of stability analysis is equivalent in mathematical sense with the design sensitivity analysis for optimization.

In designing slender or thin-walled structures, safety in view of buckling is very important. Let Λ^* denote the specified lower bound for Λ^c that is defined considering the safety factor for the design load $\boldsymbol{P}^0$. An optimum design problem for specified buckling load factor may be simply formulated as

$$\text{minimize} \quad \tilde{C}(\boldsymbol{b}), \tag{10}$$

$$\text{subject to} \quad \tilde{\Lambda}^c(\boldsymbol{b}) \geqslant \Lambda^*, \tag{11}$$

where $\tilde{C}(\boldsymbol{b})$ is the objective function such as total structural volume defined by the design variables.

In order to find the optimal solution, sensitivity coefficients of $\tilde{\Lambda}^c(\boldsymbol{b})$ with respect to a symmetric design modification is needed if a gradient-based algorithm is used. A symmetric design modification corresponds to a minor imperfection that is defined by $\bar{S}_{i\xi}\phi_i^1 = 0$ (Roorda, 1968). In the following, sensitivity equations with respect to the design parameter ξ is discussed for a vector $\boldsymbol{b}^s$ satisfying a symmetry property.

Consider a case as illustrated in Fig. 4 where the design defined by $\xi = \xi_a$ has a degenerate critical point. In this case, the critical load factor Λ^c is equal to Λ_a in Fig. 4. If ξ is increased slightly by $\Delta\xi$, λ^1 becomes positive at $\Lambda = \Lambda_a$, and Λ^c jumps to Λ_b. Note that the critical point may simply disappear. If ξ is decreased by $\Delta\xi$, Λ^c moves to Λ_d that is in the neighborhood of Λ_a. Therefore, Λ^c may be a discontinuous function of ξ for the case where a degenerate critical point exists.

Functions of ξ are denoted with a hat as $\hat{\Lambda}^c(\xi)$, $\hat{\lambda}^1(\hat{\Lambda}^c(\xi), \xi)$. Total differentiation with respect to ξ is indicated by $(\ )'$. The following relation is derived at the critical point by differentiating $\lambda^1 = 0$:

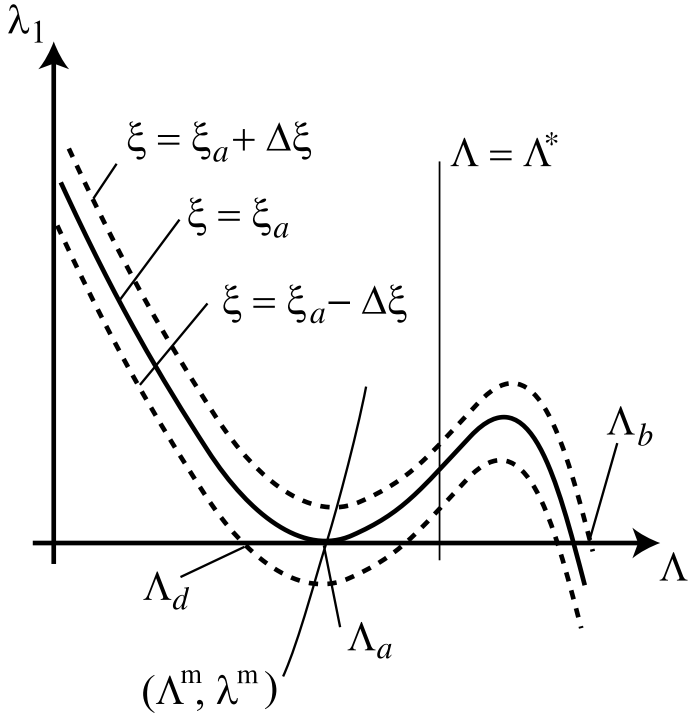

Fig. 4. Critical point for perfect and imperfect systems.

$$\lambda^1_\xi + \lambda^1_\Lambda \hat{\Lambda}^{c\prime} = 0, \tag{12}$$

where $(\ )_\xi$ indicates partial differentiation with respect to ξ while Λ is fixed. If the critical point is simple and discrete, $\hat{\Lambda}^{c\prime}$ can be computed from Eq. (12) because $\lambda^1_\Lambda \neq 0$.

Consider a case with a degenerate critical point where $\lambda^1_\Lambda = 0$ is satisfied. Computation of the sensitivity coefficients with respect to a minor imperfection or a symmetric design modification is classified as follows:

(1) $\lambda^1_\xi = W_{ij\xi}\phi^1_i\phi^1_j \neq 0$: In this case, design modification has direct effect on λ^1, and it is immediately seen from Eq. (12) that $\hat{\Lambda}^{c\prime}$ is not bounded.

(2) $\lambda^1_\xi = W_{ij\xi}\phi^1_i\phi^1_j = 0$: In this case, design modification does not have direct effect on λ^1. This type of imperfection is called *strongly minor imperfection*, in this paper, and happens in a very limited situation. For a strongly minor imperfection, $\hat{\Lambda}^{c\prime}$ is determined by the following equation which is derived by differentiating Eq. (12) again with respect to ξ:

$$\lambda^1_{\xi\xi} + 2\lambda^1_{\xi\Lambda}\hat{\Lambda}^{c\prime} + \lambda^1_{\Lambda\Lambda}(\hat{\Lambda}^{c\prime})^2 = 0, \tag{13}$$

where $\lambda^1_\Lambda = 0$ has been used. Since we consider a degenerate critical point, $\lambda^1_{\Lambda\Lambda} > 0$ holds as is seen from Fig. 1. The value of $\hat{\Lambda}^{c\prime}$ depends on $\lambda^1_{\xi\xi}$ as follows:

(a) $\lambda^1_{\xi\xi} \neq 0$: $\hat{\Lambda}^{c}$ separates to two points if $(\lambda^1_{\xi\Lambda})^2 - \lambda^1_{\xi\xi}\lambda^1_{\Lambda\Lambda} > 0$ as illustrated in points B and C in Fig. 5, and otherwise the critical point vanishes or jumps discontinuously.

(b) $\lambda^1_{\xi\xi} = 0$: The following two solutions are derived from Eq. (13):

$$\hat{\Lambda}^{c\prime} = 0 \text{ or } -2\lambda^1_{\xi\Lambda}/\lambda^1_{\Lambda\Lambda}. \tag{14}$$

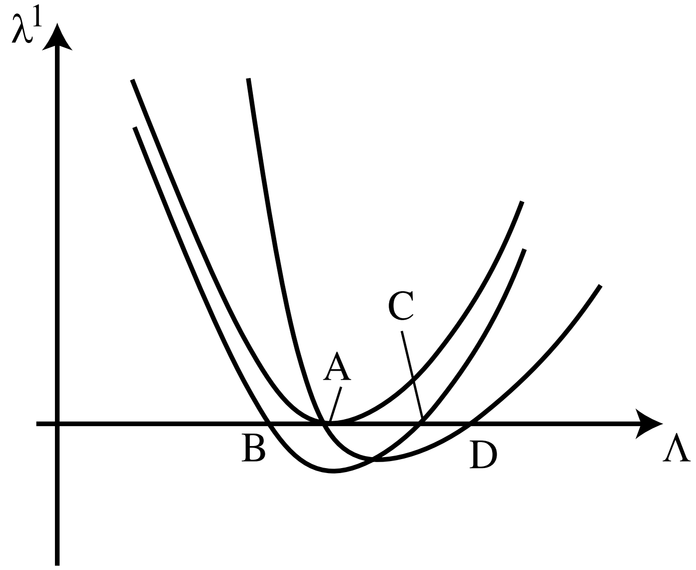

Fig. 5. Dependence of Λ^{c} on ξ.

If $\lambda^1_{\xi\Lambda} = 0$ then $\hat{\Lambda}^c$ remains the same value as ξ is varied, and otherwise the critical point separates to two points as illustrated in points A and D in Fig. 5.

Although the major imperfection where $S_{i\xi}\phi^{c1}_i \neq 0$ is satisfied is out of scope of this paper, the results for that case is briefly summarized as follows. In this case, the well known asymptotic forms of critical load factor of imperfect system are

$$\hat{\Lambda}(\xi) = \hat{\Lambda}(0) \pm \beta_1 \frac{(\beta_2\xi)^{1/2}}{W_{ij\Lambda}\phi^1_i\phi^1_j} \tag{15}$$

for an asymmetric bifurcation point, and

$$\hat{\Lambda}(\xi) = \hat{\Lambda}(0) - \gamma_1 \frac{(\gamma_2\xi)^{2/3}}{W_{ij\Lambda}\phi^1_i\phi^1_j} \tag{16}$$

for a symmetric bifurcation point, where β_1, β_2, γ_1 and γ_2 are defined by the values at the critical point of the perfect system and are independent of ξ. Therefore, even the coefficients for the asymptotic form are unbounded at a degenerate critical point satisfying $W_{ij\Lambda}\phi^1_i\phi^1_j = 0$.

We have shown that Λ^c is a discontinuous function of ξ and the sensitivity coefficient of Λ^c with respect to ξ is not bounded even for a minor imperfection. Therefore, formulation (11) is not valid as the constraint of an optimization problem to be solved by a gradient-based approach. The optimization problem can be alternatively written as

$$\text{minimize} \quad C \tag{17}$$

$$\text{subject to} \quad \lambda^1 \geqslant 0 \text{ for } 0 \leqslant \Lambda \leqslant \Lambda^*. \tag{18}$$

Let Λ^m denote the load factor at which λ^1 takes the minimum before Λ reaches Λ^*. If λ^1 is a monotonically decreasing function of Λ, λ^1 has the minimum value at $\Lambda = \Lambda^*$ in the prescribed range $0 \leqslant \Lambda \leqslant \Lambda^*$. In this case, the sensitivity coefficient of λ^1 with respect to ξ at $\Lambda = \Lambda^*$ is needed in the optimization process, and it is a continuous function of ξ and bounded. Therefore, the following discussion is restricted only to the case of $\Lambda^{\rm m} < \Lambda^*$; i.e., Λ^* is located as illustrated in Fig. 4.

It may be seen from Fig. 4 that the minimum value of λ^1 and corresponding load factor $\Lambda^{\rm m}$ along the fundamental equilibrium path is a continuous function of ξ even at the degenerate critical point. Sensitivity of $\hat{\lambda}^{\rm m} = \hat{\lambda}^1(\hat{\Lambda}^{\rm m}(\xi), \xi)$ with respect to ξ is written as

$$\hat{\lambda}^{\rm m\prime} = \lambda^1_{\xi} + \lambda^1_{\Lambda}\hat{\Lambda}^{\rm m\prime}. \tag{19}$$

Since $\lambda^1_{\Lambda} = 0$, the second term in the right-hand side of Eq. (19) vanishes, and $\hat{\lambda}^{\rm m\prime}$ is simply written as

$$\hat{\lambda}^{\rm m\prime} = \lambda^1_{\xi}. \tag{20}$$

By differentiating Eq. (5) with respect to ξ,

$$W_{ij\xi}\phi^1_j + W_{ij}\phi^1_{j\xi} = \lambda^1_{\xi}\phi^1_i + \lambda^1\phi^1_{i\xi} \tag{21}$$

and by multiplying ϕ^1_i to Eq. (21), taking summation over i, and using Eqs. (2) and (3),

$$\lambda^1_{\xi} = W_{ij\xi}\phi^1_i\phi^1_j \tag{22}$$

holds for a fixed value of Λ. Note that Eq. (22) is satisfied at any equilibrium state. The sensitivity of ϕ^1_i is not needed for computing sensitivity of λ^1 for fixed value of Λ, which is similar with the case of linear buckling loads (Haug et al., 1986). Note that the change of the load factor at which λ^1 has the minimum does not have any effect on the sensitivity of $\hat{\lambda}^{\rm m}$.

Eq. (22) is rewritten in terms of Π^S as

$$\lambda^1_{\xi} = S_{ij\xi}\phi^1_i\phi^1_j + S_{ijk}\hat{Q}'_k\phi^1_i\phi^1_j. \tag{23}$$

Note that $\hat{Q}'_k$ is needed for obtaining λ^1_{ξ}. Differentiating Eq. (1), multiplying ϕ^r_j, taking summation over j, and using Eq. (2), the following equation is derived:

$$\lambda^r\hat{Q}'_i\phi^r_i + S_{i\xi}\phi^r_i = 0, \tag{24}$$

where Λ is fixed and S_{ij} is assumed to be independent of Λ. Let U_r denote the generalized displacement in the direction of ϕ^r_i of the initial *perfect* system; i.e.,

$$\hat{Q}'_i = \sum_{r=1}^{n}\phi^r_i\hat{U}'_r. \tag{25}$$

Note that ϕ^r_i is used only for transformation of the displacements, and does not depend on ξ (Ohsaki and Uetani, 1996). By incorporating the inverse form

$$\hat{U}'_i = \hat{Q}'_i\phi^r_i \tag{26}$$

of Eq. (25) into Eq. (24), and using Eq. (3),

$$\hat{U}'_r = -\frac{S_{i\xi}\phi^r_i}{\lambda^r} \tag{27}$$

is derived.

Consider a symmetric bifurcation point where $S_{ijk}\phi^1_i\phi^1_j\phi^1_k = 0$. From Eqs. (23) and (27),

$$\lambda^1_{\xi} = S_{ij\xi}\phi^1_i\phi^1_j - \sum_{r=2}^{n}\frac{S_{ijk}\phi^1_i\phi^1_j\phi^r_k S_{l\xi}\phi^r_l}{\lambda^r} \tag{28}$$

is obtained. Note that Eq. (28) is valid also for a minor imperfection of an asymmetric bifurcation point, where the displacements are symmetric and the sensitivity $\hat{U}'_1$ does not have components in the direction of

ϕ_i^1. This way, the sensitivity coefficients of the minimum value of the lowest stability coefficient along the fundamental equilibrium path can be obtained by incorporating Eq. (28) into Eq. (20), and the optimum solution under stability constraint may be found by using a gradient-based algorithm.

4. Illustrative example

Consider a plane four-bar truss as shown in Fig. 6 with center node a and four supports in a same plane. A proportional load $\Lambda \boldsymbol{P}$ is applied at node a. The cross-sectional areas are 0.125 cm^2 for members 1 and 2, and 1.0 cm^2 for members 3 and 4; i.e., an arch-type two-bar truss is suspended by two slender cables. The elastic modulus is 205.8 GPa, and $\boldsymbol{P} = 9.8$ kN. The geometrical parameter L is 100 cm. The total Lagrangian formulation and displacement increment method with error correction at each step have been used for tracing the fundamental equilibrium path. The members are assumed to be in the elastic range. The truss has a bifurcation point for a large value of H_1, and has a limit point if H_1 is small. In the following example, H_1 is fixed at 320 cm.

The displacement v in the negative y-direction of node a is plotted with respect to Λ as shown in Fig. 7 for $H_2 = 170$, 180, 190 cm. Variation of λ^1 is as shown in Fig. 8. It is observed from Fig. 8 that a degenerate critical point exists at $H_2 \simeq 180$ cm. Note that Λ increases in the vicinity of the critical point which is a bifurcation point corresponding to an antisymmetric buckling mode where the node a moves in the x-direction. The degenerate critical point is divided to two critical points if $H_2 = 190$ cm, and vanishes for the case of $H_2 = 170$ cm.

Consider an imperfect system where node a is moved in the x-direction by 0.5 cm, where $H_2 = 180$ cm^2. Variation of the displacement u in the x-direction of node a is plotted in Fig. 9 with respect to Λ. It is seen

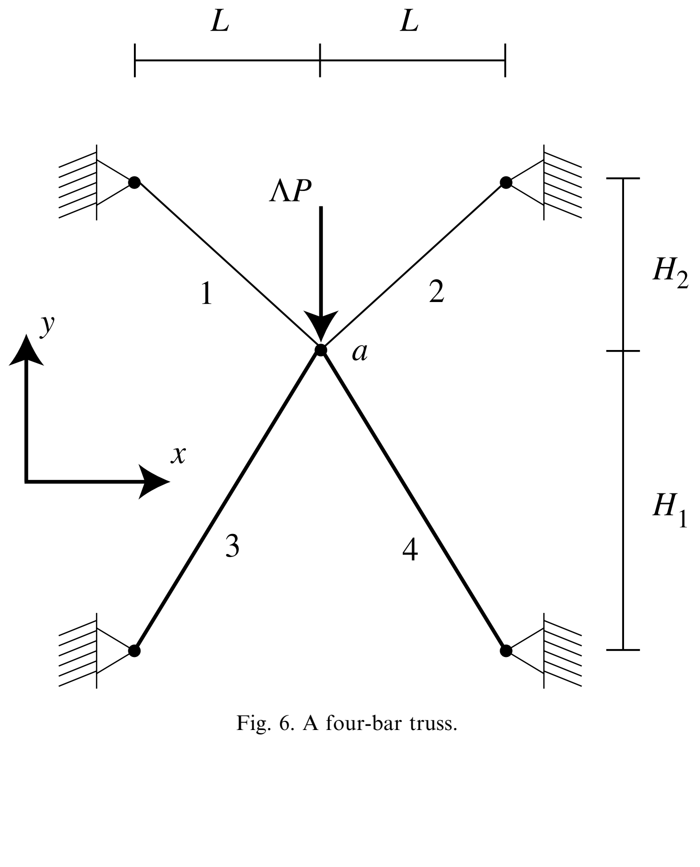

Fig. 6. A four-bar truss.

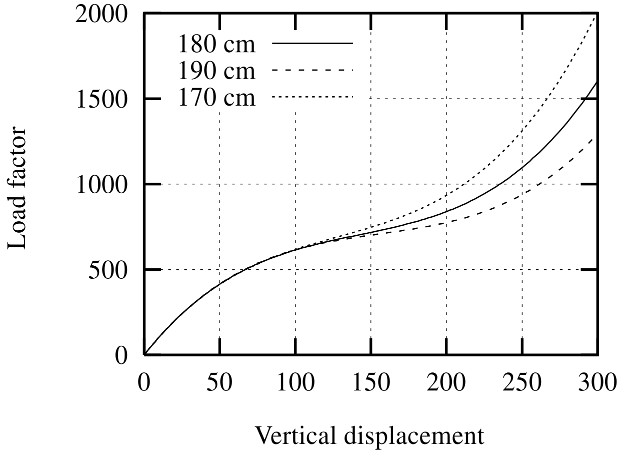

Fig. 7. Relation between load factor and displacements.

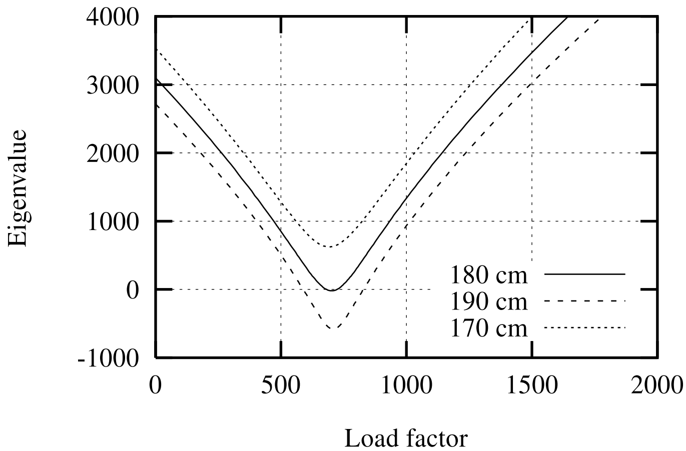

Fig. 8. Relation between load factor and lowest eigenvalue.

from Fig. 9 that u increases in the neighborhood of the degenerate critical point, but it decreases immediately to a small value. Note that u is slightly reversed along the equilibrium state beyond the critical point, and $\lambda^1 > 0$ is satisfied along the equilibrium path. This reversal is not due to numerical error, because almost same equilibrium path has been traced if a very small displacement increment has been used. Relation between the horizontal and vertical displacements is plotted in Fig. 10. Note that a very smooth curve is observed in this figure.

Sensitivity analysis of λ_1 has been carried out for a modification of the cross-sectional area A^s of suspending members 1 and 2. The minimum value of λ^1 along the fundamental equilibrium path for $A^s = 0.1261\,\mathrm{cm}^2$ is -0.75953, which may be considered to be almost equal to 0 because the initial value at the undeformed state is 3100.9. Sensitivity coefficient of the lowest eigenvalue by Eq. (28) is 17 060. The minimum value of λ^1 for $A^s = 0.1251\,\mathrm{cm}^2$ is -18.157, and the sensitivity coefficient by finite difference approximation is 17 398. The error in the sensitivity coefficient is due to the approximation in defining the minimum value of λ^1 from the discrete data along the fundamental equilibrium path.

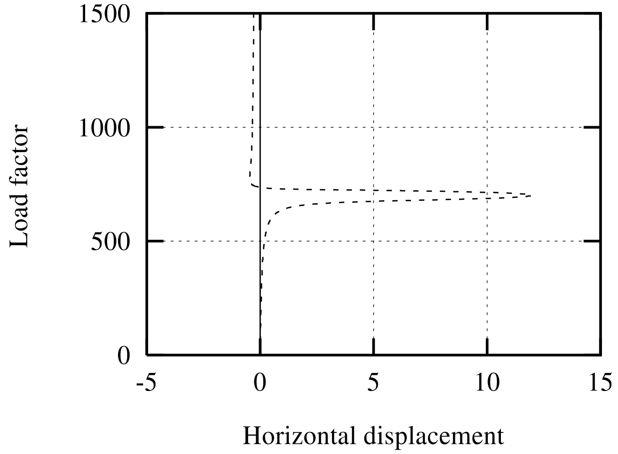

Fig. 9. Relation between horizontal displacement and Λ of an imperfect system.

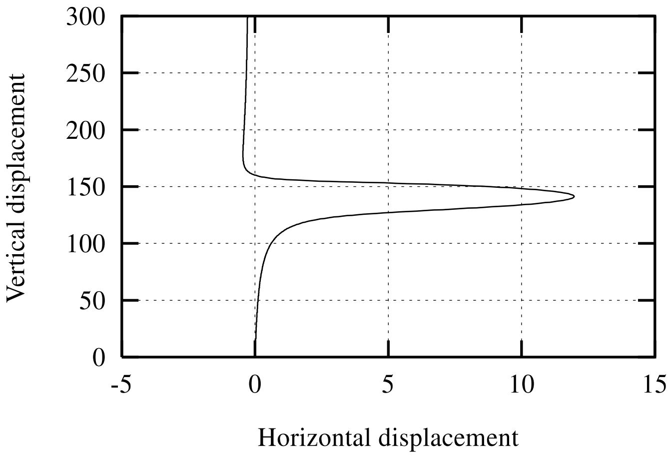

Fig. 10. Relation between horizontal and vertical displacements of an imperfect system.

Consider a simple optimization problem where A^s is considered as design variable. The parameter H_2 is fixed at 180 cm^2. Variation of Λ^c is plotted in Fig. 11 with respect to A^s. The objective function is A^s, and the lower bound Λ^* for Λ^c is 1000.0. It is observed from Fig. 11 that Λ^c increases as A^s is increased from 0.12 cm^2. The truss has a degenerate critical point around $A^s = 0.1261$ cm^2, and the critical point disappears as A^s is further increased. Therefore, the constraint $\Lambda^c \geqslant 1000$ is not satisfied if $A^s < 0.1261$ cm^2, and Λ^c is a discontinuous function of A^s. Since the lowest eigenvalue, however, is a continuous function of A^s, the optimal value of A^s, which is slightly above 0.1261 cm^2 may be successfully found by the proposed formulation and the method of sensitivity analysis.

5. Conclusions

An analytical formulation has been presented for sensitivity analysis of minimum value of the lowest eigenvalue along the fundamental equilibrium path of an elastic structure. The sensitivity coefficient of the

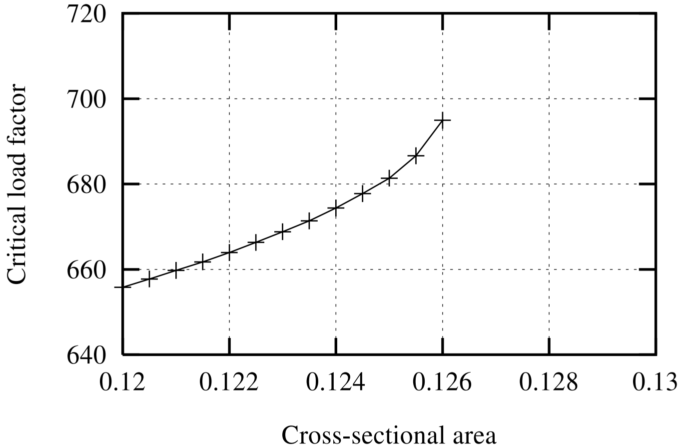

Fig. 11. Variation of critical load factor with respect to cross-sectional area.

lowest eigenvalue is easily computed for a fixed value of the load coefficient, and the change in the load factor at which the eigenvalue takes the minimum does not have any effect on the sensitivity coefficient of the minimum value of the eigenvalue.

It has been shown that sensitivity of the degenerate critical load factor is not bounded even for a minor imperfection. Therefore, a simple formulation of optimum design problem under the lowest eigenvalue is not valid for the case with degenerate critical point, because the critical load factor is a discontinuous function of the design parameter.

A new formulation has been presented for optimum design under constraint on the minimum value of the lowest eigenvalue along the fundamental equilibrium path before reaching the specified load factor. The accuracy of the proposed formula of sensitivity analysis of the minimum eigenvalue is discussed through an example of a simple four-bar truss. Optimal cross-sectional areas of a finite dimensional structure that may have a degenerate critical point can be successfully obtained by using the proposed method of sensitivity analysis.

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